DOI:
10.1007/s00220-003-1014-2

Document Version
Early version, also known as pre-print

Link to publication record in King's Research Portal

Citation for published version (APA):
Apostolov, V., & Salamon, S. (2004). Kaehler reduction of metrics with holonomy G_2. COMMUNICATIONS IN MATHEMATICAL PHYSICS, 246(1), 43-61. 10.1007/s00220-003-1014-2

# KÄHLER REDUCTION OF METRICS WITH HOLONOMY $G_2$

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## Introduction

There are now many explicitly known examples of metrics with holonomy group equal to $G_2$, the simplest of which admit an isometry group with orbits of codimension one. A metric with holonomy $G_2$ on a smooth 7-manifold $Y$ is characterized by a 3-form and a 4-form that are interrelated and both closed. If this structure is preserved by a circle group $S^1$ acting on $Y$ (which cannot then be compact), the quotient $Y/S^1$ has a natural symplectic structure. A geometrical description of such quotients was carried out by Atiyah and Witten [8] when $Y$ is one of three original manifolds with a complete metric of holonomy $G_2$ described in [14]. The symplectic structure is also defined on the image of the fixed points, which in each of these cases is a Lagrangian submanifold $L$. The embedding of $L$ in a neighborhood of $Y/S^1$ is believed to approximate the geometry of a special Lagrangian submanifold of a Calabi-Yau manifold, and this is consistent with models of special Lagrangian submanifolds of $\mathbb{R}^6$ described for example in [24].

This work motivates a general investigation of the quotient $N = Y/S^1$ of a manifold with holonomy $G_2$ by an $S^1$ action. Initial results in this direction can be found in [15, 16], and in this paper we pursue the theory under the simplifying assumption that the $S^1$ action is free. It is an elementary fact that $N$ has, in addition to its symplectic 2-form, a natural reduction to $SU(3)$. Whilst this structure cannot be torsion-free (i.e. Calabi-Yau) if $Y$ is irreducible, there are non-trivial examples in which the associated almost-complex structure is integrable, so that $N$ is Kähler. This paper is devoted to an investigation of such a situation, which turns out to be surprisingly rich. Our study began with the realization that in this case, a parameter measuring the size of the fibers of $Y$ generates both a Killing and a Ricci potential for $N$ (Proposition 1 and Corollary 2), exhibiting a link with the theory of the so-called Hamiltonian 2-forms [4].

We first describe the induced $SU(3)$ structure on $N$ in §1, and then pursue consequences of the integrability condition. In particular, we prove that when $N$ is Kähler, an infinitesimal isometry $U$ of the $SU(3)$ structure is inherently defined. The situation is reminiscent of the study of Einstein-Hermitian 4-manifolds in which Killing vector fields appear automatically [6, 17, 27]. We explain in §2 that $U$ can be used to obtain a Kähler quotient of $N$, consisting of a 4-manifold equipped with a 1-parameter family of smooth functions and 2-forms, satisfying a coupled second-order evolution equation (Theorem 1). The procedure can be reversed so as to construct metrics with holonomy (generically equal to) $G_2$ from a 4-manifold $M$ with the appropriate structure.

A fundamental construction of a holonomy $G_2$ metric starting from a $\mathbb{T}^2$ bundle over a hyperkähler 4-manifold was discovered by Gibbons, Lü, Pope and Stelle [20], and towards the end of §3 we exhibit our reverse procedure as a generalization of the above. It can be further improved by introducing an anti-self-dual 2-form with the opposite orientation to the Kähler one (Theorem 2). This leads to the construction of new examples of $G_2$ metrics based on the examples of Ricci-flat almost-Kähler metrics [5, 7, 28]. We point out that there also exist constructions of holonomy $G_2$ metrics from higher dimensional hyperkähler manifolds (see for example [3]).

When $M$ is $\mathbb{T}^4$, the basic examples are modelled on nilpotent Lie groups and fall into three types, special cases of which were also mentioned in [15]. We explain how to show that these are irreducible and therefore have holonomy equal to $G_2$. In §4, we analyse the holonomy of these metrics by restricting them to hypersurfaces so as to induce an $SU(3)$ structure that evolves according to equations studied by Hitchin [21]. Our examples provide perhaps the simplest instance of this phenomenon other than the case of nearly-Kähler manifolds. The different $SU(3)$ structures obtained in this way, an integrable one on a quotient and non-integrable ones on hypersurfaces, are naturally linked via the common 7-manifold.

In the final section, we study a more general class of solutions of our system, formulated in terms of the complex Monge-Ampère equation. This leads to both an abstract existence theorem, some special solutions, and an explicit final example.

Our assumption that the symplectic manifold $Y/S^1$ be Kähler leads to a (local) action on $Y$ not just by $S^1$ but by the torus $\mathbb{T}^2$ (Corollary 3), though examples in §4 exhibit $\mathbb{T}^2$ actions for which the Kähler assumption fails (see also [11, 30]). We hope nonetheless that similar methods will lead to the classification of metrics with holonomy $G_2$ admitting a $\mathbb{T}^2$ action.

**Acknowledgments.** This material was first developed at the joint AMS–UMI conference in Pisa in 2002. The first author was supported in part by NSERC grant OGP0023879 and by NSF grant INT-9903302. The second author is a member of EDGE, Research Training Network HPRN-CT-2000-00101, supported by the European Human Potential Programme, and a preliminary version of the results was presented at its mid-term conference. The authors are grateful to the referee for useful remarks, and to D. Conti for subsequent checking of some curvature computations.

## 1. The first reduction

In this section we study general properties of a 7-dimensional Riemannian manifold $(Y,k)$ endowed with a torsion-free $G_2$ structure and an infinitesimal isometry $V$. The $G_2$ structure on $Y$ is defined by an admissible 3-form $\varphi$, which itself determines the Riemannian metric $k$, an orientation and Hodge operator $*$. The torsion-free condition is equivalent to the closure of both $\varphi$ and the 4-form $*\varphi$, and amounts to asserting that the holonomy group of $k$ is contained in $G_2$. This theory is elaborated in the standard references [13, 19, 25, 29]

We denote by $(N,\check{k})$ the Riemannian quotient of $(Y,k)$, so that $N$ is a 6-dimensional manifold formed from the orbits of the Killing vector field $V$. The considerations are local and hold in a suitable neighborhood of any point of $Y$ where the vector field $V$ is non-zero. The Riemannian metric $\check{k}$ is induced by the formula

$$k = \check{k} + t^{-2}\eta \otimes \eta,$$

where $t = \|V\|_k^{-1} = k(V,V)^{-1/2}$ and $\eta = k(\,\cdot\,, t^2 V)$ is the 1-form dual to $t^2V$ so that $\eta(V) = 1$. In the case when $Y$ can be realized as a principal $S^1$ bundle over $N$ (the fibers being the closed orbits of $V$), $\eta$ is nothing but a connection 1-form.

Since $V$ is a Killing vector field, $dt(V) = 0$ and $t$ descends to a function on $N$. The 2-form

$$\sigma = \imath_V \varphi$$

is horizontal in the sense that $\imath_V\sigma = 0$, and since $V$ preserves the $G_2$ structure we have

$$d\sigma = d(\imath_V\varphi) = \mathcal{L}_V\varphi = 0,$$
$$\mathcal{L}_V\sigma = d(\imath_V\sigma) = 0.$$

Thus, $\sigma$ is closed and $V$-invariant, and therefore the pullback of a closed 2-form on $N$, again denoted by $\sigma$. (We identify functions and forms on $N$ with their pullbacks to $Y$ throughout, and $\mathcal{L}$ and $\imath$ stand for the Lie derivative and the interior product.) Since $\sigma$ is non-degenerate transverse to the fibers of $Y$, it defines a symplectic form on $N$.

We now choose to express the forms characterizing the $G_2$ structure on $Y$ as

$$\begin{aligned} \varphi &= \sigma\wedge\eta + t\Psi^+ \\ *\varphi &= \Psi^-\wedge\eta + \tfrac{1}{2}t^2\sigma\wedge\sigma, \end{aligned} \tag{1}$$

where

$$\Psi^- = -\imath_V(*\varphi).$$

If $\eta$ were replaced by the unit form $t^{-1}\eta$ then both $\sigma$ and $\Psi^-$ would need to be multiplied by a compensatory factor of $t$ if the left-hand sides of (1) are to remain the same. This explains why $\sigma\wedge\sigma$ appears in (1) with a coefficient of (one half) $t^2$, and $\Psi^+$ appears with a coefficient of $t$ so as to have the same norm as $\Psi^-$. The rescaled Riemannian metric

$$h = t^{-1}\check{k} \tag{2}$$

is compatible in the sense that the skew-symmetric endomorphism $J$ defined by

$$h(J\,\cdot\,,\,\cdot\,) = \sigma(\,\cdot\,,\,\cdot\,) \tag{3}$$

is an almost-complex structure on $N$. The triple $(h, \sigma, J)$ then defines an *almost-Kähler* structure on $N$, though the qualification 'almost' can be deleted when $J$ is integrable.

Just as for $\sigma$, the 3-form $\Psi^-$ is closed and basic (meaning $\imath_V\Psi^- = 0$, $\mathcal{L}_V\Psi^- = 0$). Thus, it too is the pullback of a real form on $N$. This has type $(3,0)+(0,3)$ with respect to $J$, so that $\Psi^-(J\,\cdot\,, J\,\cdot\,,\,\cdot\,) = -\Psi^-(\,\cdot\,,\,\cdot\,,\,\cdot\,)$. The theory of $G_2$ structures then implies that

$$\Psi^+(\,\cdot\,,\,\cdot\,,\,\cdot\,) = \Psi^-(J\,\cdot\,,\,\cdot\,,\,\cdot\,), \tag{4}$$

and the two real 3-forms combine to define a complex form

$$\Psi = \Psi^+ + i\Psi^- \tag{5}$$

of type $(3,0)$ with respect to $J$.

Unlike $\Psi^-$, the 3-form $\Psi^+$ is not in general closed; indeed, $d\Psi^+$ can be identified with the Nijenhius tensor of $J$, the obstruction to the integrability of $J$:

**Lemma 1.** *$\Psi^+$ is closed if and only if the almost-complex structure $J$ is integrable.*

*Proof.* Since $\Psi^-$ is already closed, the exterior derivative $d\Psi^+$ of (5) is real. But if $J$ is integrable so that $(N,J)$ is a complex manifold, this real 4-form has type $(3,1)$ and therefore vanishes. Conversely, if $d\Psi^+ = d\Psi^- = 0$ then writing (5) locally as a wedge product of $(1,0)$-forms $\alpha^i$ shows that $(d\alpha^i)^{0,2} = 0$ for $i = 1,2,3$, and $J$ is integrable. □

The almost-Kähler structure corresponds to a reduction to $U(3)$ at each point of $N$. Specification of the non-zero element (5) in the space $\Lambda^{3,0}$ is precisely what is needed to reduce the structure to $SU(3)$, but one should first rescale given that $\|\Psi^\pm\|_h$ is not in general constant. The 3-forms

$$\psi^\pm = t^{-1/2}\Psi^\pm$$

have norm equal to 2. They are subject to the compatibility equations

$$\sigma \wedge \psi^\pm = 0, \tag{6}$$

and

$$\psi^+ \wedge \psi^- = \tfrac{2}{3}\sigma^3 = 4\mathrm{vol}_h, \tag{7}$$

consistent with (3), and give rise to an $SU(3)$ structure with underlying metric $h$. Notation for the 3-forms is now consistent with that of [15]. Since $\psi^+ + i\psi^- \in \Lambda^{3,0}$, we have

$$(\gamma + iJ\gamma) \wedge (\psi^+ + i\psi^-) = 0$$

for any 1-form $\gamma$, where $J$ acts on 1-forms by $J\gamma(\,\cdot\,) = -\gamma(J\,\cdot\,)$. Whence

$$\gamma \wedge \psi^+ = J\gamma \wedge \psi^-. \tag{8}$$

This last equation will be useful for calculations.

The real 3-forms $\psi^+, \psi^-$ have a common stabilizer group $SL(3,\mathbb{C})$ at each point of $N$, and either one determines the almost-complex structure $J$ [22]. From this point of view, the further reduction to $SU(3)$ is achieved by a 2-form $\sigma \in \Lambda^{1,1}$ such that $\sigma(\,\cdot\,, J\,\cdot\,) = h(\,\cdot\,,\,\cdot\,)$ is positive definite. The $SU(3)$ structure is therefore fully determined by (say) $\psi^+$ and $\sigma$; in the new notation Lemma 1 reads as

**Corollary 1.** *The almost-complex structure $J$ is integrable if and only if*

$$\nabla\psi^+ = -\tfrac{1}{2}(d^c \log t) \otimes \psi^-, \quad \nabla\psi^- = \tfrac{1}{2}(d^c \log t) \otimes \psi^+,$$

*where $\nabla$ denotes the Levi-Civita connection of $h$ and $d^c f = J df$.*

*Proof.* Given the above equations,

$$d\Psi^+ = d(t^{1/2}\psi^+) = \tfrac{1}{2}t^{1/2}(d\log t \wedge \psi^+ - d^c \log t \wedge \psi^-),$$

but the right-hand side is zero by (8). Thus, $J$ is integrable by Lemma 1.

Conversely, if $J$ is integrable then $(h, J)$ is Kähler and $\nabla_U \psi^{\pm}$ is a form of type $(3,0)+(0,3)$ for any vector field $U$. Since $\psi^+$ and $\psi^-$ are mutually orthogonal and of constant norm 2,

$$\nabla\psi^+ = \gamma \otimes \psi^-, \quad \nabla\psi^- = -\gamma \otimes \psi^+$$

for some 1-form $\gamma$. Using (8) again, we obtain

$$\begin{aligned} d\Psi^+ &= d(t^{1/2}\psi^+) = \tfrac{1}{2}t^{1/2}(d\log t \wedge \psi^+ + 2\gamma \wedge \psi^-) \\ &= \tfrac{1}{2}t^{1/2}(-2J\gamma + d\log t) \wedge \psi^+. \end{aligned}$$

The claim follows by Lemma 1 and the injectivity of the linear map $\Lambda^1 \to \Lambda^4$ defined at each point by wedging with $\psi^+$. □

**Corollary 2.** *If $J$ is integrable, the Ricci form $\kappa$ of the Kähler manifold $(N, h, J)$ is given by*

$$\kappa = \tfrac{1}{2} dd^c \log t.$$

*Proof.* This is again an immediate consequence of the integrability criterion, which implies that the $(3,0)$-form $\Psi = \Psi^+ + i\Psi^-$ is closed. It follows that $\Psi$ is a holomorphic section of the canonical bundle $\Lambda^{3,0}$ and the Ricci form is given by

$$\kappa = i\partial\overline{\partial}\log(\|\Psi\|_h^2) = \tfrac{1}{2}dd^c\log(\|\Psi\|_h^2),$$

and the result follows from (7). □

In the case when $J$ is integrable, it is now possible to formulate a complete system of conditions for the $SU(3)$ structure on $N$ to arise from the quotient of a torsion-free $G_2$ structure.

**Proposition 1.** *Let $(N, h, J, \sigma)$ be a Kähler manifold of real dimension 6, endowed with a compatible $SU(3)$ structure defined by a $(3,0)$-form $\psi^+ + i\psi^-$. Then this structure is obtained as the quotient of a 7-dimensional manifold $Y$ with a torsion-free $G_2$ structure by a nontrivial infinitesimal isometry $V$ if and only if*

(i) $d\psi^+ = -\frac{1}{2}(d^c \log t) \wedge \psi^-$ *and* $d\psi^- = \frac{1}{2}(d^c\log t) \wedge \psi^+$ *for a smooth positive function $t$, and*

(ii) *the Hamiltonian vector field $U$ on $(N, \sigma)$, corresponding to $-t$, is an infinitesimal isometry of the $SU(3)$ structure.*

*In this case, the corresponding 7-manifold $Y$ is locally $\mathbb{R} \times N$ and the metric $k$, infinitesimal isometry $V$ and $G_2$ invariant forms $\varphi, *\varphi$ are given by*

$$\begin{gathered} k = th + t^{-2}\eta \otimes \eta, \qquad V = \frac{\partial}{\partial y} \\ \varphi = \sigma \wedge \eta + t^{3/2}\psi^+, \qquad *\varphi = t^{1/2}\psi^- \wedge \eta + \tfrac{1}{2}t^2 \sigma \wedge \sigma, \end{gathered} \tag{9}$$

*where $y$ is a variable for the $\mathbb{R}$ factor, and $\eta = dy + \eta_N$ is a 1-form on $\mathbb{R} \times N$ for which*

$$d\eta_N = -t^{1/2}(\imath_U \psi^+). \tag{10}$$

*Proof.* Condition (i) is necessary by Corollary 1, and the equalities (9) reflect the earlier definitions of $h, \sigma, \psi^{\pm}$. Moreover, $\partial/\partial y$ is identified with the Killing field $V$ (so that $k(V,V) = t^{-2}$) and $\eta$ is the corresponding connection form. Using (i) and (8) in that order, we obtain

$$\begin{aligned} 0 \;=\; d\varphi \;&=\; \sigma \wedge d\eta + \tfrac{3}{2}t^{1/2}dt \wedge \psi^+ - \tfrac{1}{2}t^{1/2}Jdt \wedge \psi^- \\ &=\; \sigma \wedge d\eta + t^{1/2}dt \wedge \psi^+. \end{aligned}$$

Consequently, $(d\eta)^{1,1} = 0$ and (10) follows from the fact that $\imath_U \sigma = -dt$.

We now have

$$0 = -d(d\eta) = d(t^{1/2}\imath_U\psi^+) = d(\imath_U\Psi^+) = \mathcal{L}_U\Psi^+ = t^{1/2}\mathcal{L}_U\psi^+.$$

Since $U$ is Hamiltonian, we also have

$$\mathcal{L}_U\sigma = 0,$$

i.e. $U$ is an infinitesimal isometry for the pair $(\sigma, \psi^+)$, and therefore for the $SU(3)$ structure (using [22]). Reversing the above arguments, one can check directly that (9) and (10) define a torsion-free $G_2$ structure on $Y = \mathbb{R} \times N$. □

**Corollary 3.** *Under the hypothesis of Proposition 1, the horizontal lift of $U$ to $(Y, k)$ is an infinitesimal isometry of the $G_2$ structure $\varphi$, which commutes with $V$.*

*Proof.* This is an immediate consequence from Proposition 1. □

**Remark 1.** It follows from Proposition 1 that when $t$ is constant, $(Y, k)$ is locally the Riemannian product of a Calabi-Yau 6-manifold with $\mathbb{R}$. In this case the holonomy group of $h$ lies in $SU(3)$. In general, the failure of the holonomy to reduce to $SU(3)$ is measured by a torsion tensor

$$\tau \in T_n^*N \otimes \frac{\mathfrak{so}(6)}{\mathfrak{su}(3)}$$

determined by $d\sigma, d\psi^+, d\psi^-$. Whereas $\tau$ has a total of 42 components, exactly two thirds of these will always vanish on $N$ and $\tau$ is determined by the remaining 14 tracefree components of $d\psi^+$, or 6 in the Kähler case [15].

## 2. A second reduction

Proposition 1 is the key ingredient for performing a further quotient via the infinitesimal isometry $U$. To carry this out, we shall assume from now on that $t$ is not constant and that $J$ is integrable. Thus, $U$ is a non-trivial infinitesimal isometry of the $SU(3)$ structure on $N$ determined by the pair $(h, \psi^+)$.

Denote by $(M, J_1)$ the 'stable' or holomorphic quotient of $(N, J)$, defined at least locally as the complex two-dimensional manifold of holomorphic leaves of the foliation generated by $\Xi = U - iJU$. Here we exploit the fact that $\Xi$ is a holomorphic vector field on $(N, J)$. Since $J$ is integrable, $\Psi = \Psi^+ + i\Psi^-$ is a closed $(3,0)$-form, and we set

$$\Omega = \tfrac{1}{2}\imath_\Xi\Psi.$$

Since $\imath_\Xi\Omega = 0$, it follows from (4) that $\Omega$ is closed and the pullback of a holomorphic symplectic form on $M$, again denoted by $\Omega$. The real closed 2-forms

$$\omega_2 = \mathfrak{Re}\,\Omega, \quad \omega_3 = \mathfrak{Im}\,\Omega$$

on $M$ (that pull back to $\imath_U\Psi^+$, $\imath_U\Psi^-$ respectively on $N$) satisfy

$$\omega_2 \wedge \omega_2 = \omega_3 \wedge \omega_3, \qquad \omega_2 \wedge \omega_3 = 0. \tag{11}$$

The complex structure $J_1$ can now be determined by the formula

$$\omega_2(\,\cdot\,,\,\cdot\,) = \omega_3(J_1\,\cdot\,,\,\cdot\,),$$

and equation (10) in Proposition 1 reads

$$d\eta = -\omega_2. \tag{12}$$

Set $u = \|U\|_h^{-2}$ and let $\xi$ be the 1-form $h$-dual to the vector field $uU$, so that $\imath_U\xi = 1$ and $\xi = uJdt$. The 3-forms $\Psi^\pm$ are completely determined by (10) which forces them to be the real and imaginary components of $(\omega_2 + i\omega_3)\wedge(\xi + iJ\xi)$. Thus

$$\begin{aligned} \psi^+ &= t^{-1/2}(\omega_2 \wedge \xi + u\omega_3 \wedge dt), \\ \psi^- &= t^{-1/2}(\omega_3 \wedge \xi - u\omega_2 \wedge dt). \end{aligned} \tag{13}$$

They are $U$-invariant in accordance with (ii) in Proposition 1.

For any regular value of the momentum map $-t$, the stable quotient $(M, J_1)$ of $(N, J)$ can be identified with the *symplectic* quotient $(M, \tilde\omega_1(t))$ of $(N, \sigma)$ generated by the vector field $U$. In this way, we obtain the *Kähler* quotient $(M, g(t), \tilde\omega_1(t), J_1)$. In this correspondence,

$$\begin{aligned} \sigma &= \tilde\omega_1(t) + dt \wedge \xi, \\ h &= g(t) + u^{-1}\xi \otimes \xi + u\,dt \otimes dt, \end{aligned} \tag{14}$$

so that the equation (7) reduces to

$$t\tilde\omega_1(t) \wedge \tilde\omega_1(t) = \tfrac{1}{2}u\,\Omega \wedge \overline{\Omega} = u\omega_2 \wedge \omega_2 = u\omega_3 \wedge \omega_3. \tag{15}$$

To ease the notation, we shall below omit the explicit dependence of $\tilde\omega_1 = \tilde\omega_1(t)$ on $t$ except on occasions for emphasis.

We now denote by $P$ the (locally defined) space of orbits of $U$, so that $N$ can be thought as an $\mathbb{R}$ bundle over $P$ with connection 1-form $\xi$. Locally, $N = \mathbb{R} \times P$, and introducing a variable $x$ for the $\mathbb{R}$ factor, we may write

$$\xi = dx + \xi_P, \qquad U = \frac{\partial}{\partial x}$$

for some 1-form $\xi_P$ on $P$. The space of orbits of the vector field $JU$ on $P$ is the stable quotient of $(N, J)$, whereas the symplectic quotients of $(N, \sigma)$ are identified with the level sets of $t$ in $P$. Using a local description $P = \mathbb{R}^+ \times M$ in which the $\mathbb{R}^+$-factor corresponds to $t$, we may regard $u$ as a function on $M$ for each value of $t$. In these terms, we have

$$d\xi = d_P\xi_P = \alpha_M \wedge dt + \beta_M,$$

$$d_P u = u'dt + d_M u,$$

$$d\tilde\omega_1 = \tilde\omega_1' \wedge dt,$$

where $\alpha_M=\alpha_M(t), \beta_M=\beta_M(t)$ are 1-parameter families of forms on $M$, $'$ denotes $\partial/\partial t$ and $d, d_P, d_M$ denote exterior derivative on $N, P, M$.

Differentiating the first relation gives

$$d_M\beta_M=0, \quad \beta_M'=-d_M\alpha_M.$$

Using the formula (14) for the symplectic form $\sigma$ yields

$$\tilde\omega_1'=\beta_M,$$

and it follows that $\beta_M$ has type $(1,1)$ relative to $J_1$. Using (13) gives

$$\begin{aligned} d\psi^+ &= -\tfrac12 t^{-3/2}dt\wedge\omega_2\wedge\xi+t^{-1/2}\omega_2\wedge d\xi+t^{-1/2}d_Mu\wedge\omega_3\wedge dt\\ &= -\tfrac12 t^{-1}Jdt\wedge\psi^-+t^{-1/2}(\omega_2\wedge\alpha_M+d_Mu\wedge\omega_3)\wedge dt, \end{aligned}$$

since $\omega_2\wedge\beta_M=0$. In view of Proposition 1(i), $\omega_2\wedge\alpha_M+d_Mu\wedge\omega_3=0$, whence

$$\alpha_M=J_1du=d_M^cu,$$

and everything can be expressed in terms of $\tilde\omega_1$ and $u$.

In summary,

**Theorem 1.** *Let $(Y,\varphi)$ be a 7-manifold with a torsion-free $G_2$ structure, admitting an infinitesimal isometry $V$. Suppose that the norm of $V$ is not constant, and let $y\in Y$ be a point where $V$ does not vanish. Let $\mathcal V$ be a neighborhood of $y$, such that the space of orbits of $V$ in $\mathcal V$ is a manifold $N$ and suppose that the almost-Kähler structure on $N$ is in fact Kähler. Then, there exists a 4-dimensional manifold $M$ endowed with a complex structure $J_1$, a complex symplectic form $\Omega=\omega_2+i\omega_3$, and 1-parameter families of Kähler 2-forms $\tilde\omega_1=\tilde\omega_1(t)$ and positive functions $u=u(t)$ on $M$, satisfying the relations*

$$\tilde\omega_1''=-d_Md_M^cu \tag{16}$$

$$t\tilde\omega_1\wedge\tilde\omega_1=\tfrac12 u\,\Omega\wedge\overline{\Omega}. \tag{17}$$

*On a sufficiently small neighborhood of $y$, $(\varphi,V)$ is equivariantly isometric to the torsion-free $G_2$ structure*

$$\begin{aligned} \varphi &= \tilde\omega_1\wedge(dy+\eta_N)+dt\wedge(dx+\xi_P)\wedge(dy+\eta_N)\\ &\qquad +t\Big(\omega_2\wedge(dx+\xi_P)+u\omega_3\wedge dt\Big), \end{aligned} \tag{18}$$

*on $\mathbb R_t^+\times\mathbb R_{x,y}^2\times M$ endowed with the infinitesimal isometry $V=\partial/\partial y$, where*

*$d_M$ denotes the differential on $M$, and $d_M^c=J_1\circ d_M$;*
*$t>0$ is the variable on the $\mathbb R_t^+$-factor;*
*$(x,y)$ are standard coordinates on $\mathbb R_{x,y}^2=\mathbb R_x\times\mathbb R_y$;*
*$\eta_N$ is a 1-form on $N=\mathbb R_t^+\times\mathbb R_x\times M$ with $d\eta_N=-\omega_2$;*
*$\xi_P$ is a 1-form on $P=\mathbb R_t^+\times M$ with $d\xi_P=(d_M^cu)\wedge dt+\tilde\omega_1'$.*

**Remark 2.** By redefining the local coordinate $x$, one can assume (without loss) that $\eta_N$ is in fact a 1-form on $M$. In this case the $G_2$ structure $\varphi$ is invariant under $\frac{\partial}{\partial x}$ as well, and $\frac{\partial}{\partial x}$ is identified with the Killing vector field defined in Corollary 3.

It is not difficult to see that for *generic* data on $M$, the holonomy group of the $G_2$ structure (18) is *equal* to $G_2$. Indeed, the general theory of holonomy groups (see e.g. [29]) implies that if the holomomy group of $(Y, k)$ were strictly less than $G_2$, then there would exist a non-trivial parallel vector field $X$ on $(Y, k)$ commuting with $V = \frac{\partial}{\partial y}$; it would therefore come from an infinitesimal isometry of the $SU(3)$ structure on $N$ (still denoted by $X$), which preserves the level sets of $t$ and commutes with $U = \frac{\partial}{\partial x}$. The equations for $d\eta_N$ and $d\xi_P$ imply that there is no parallel vector field in span$\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ unless $d_M u = 0$ and $\tilde{\omega}' = 0$ (a situation which we shall exclude below), thus showing that $X$ is in general different from $U$. Therefore, $X$ must come from a (real) holomorphic vector field on $(M, J)$ which preserves the Kähler metrics $\omega(t)$ for each $t$. In particular, if we assume that $(M, J_1, \Omega, \tilde{\omega}_1(t), u(t))$ does not admit any infinitesimal isometry and either $d_M u \neq 0$ or $\tilde{\omega}' \neq 0$, then we know that the holonomy group of $(Y, \varphi)$ must equal $G_2$.

It is important to note that the triple $(\tilde{\omega}_1(t), \omega_2, \omega_3)$ appearing in Theorem 1 does not in general constitute a hyperkähler structure. Indeed, by (17), the holomorphic section $\Omega$ of the canonical bundle $\Lambda^{2,0}M$ satisfies

$$\|\Omega\|_g^2 = tu^{-1},$$

and the Ricci form $\kappa$ of the Kähler metric $\tilde{\omega}_1(t)$ is given by

$$\kappa = \tfrac{1}{2} d_M d_M^c (\log t - \log u) = -\tfrac{1}{2} d_M d_M^c \log u. \tag{19}$$

In the next section, we shall however explain how the simplest case does correspond to a hyperkähler situation.

## 3. Constant solutions

A careful inspection of the proof of Theorem 1 shows that the process can be inverted so as to construct a torsion-free $G_2$ structure from a 4-manifold $M$ with a complex symplectic structure $(J_1, \Omega)$ together with a 1-parameter family $(\tilde{\omega}_1(t), u(t))$ of Kähler forms and smooth functions satisfying (16) and (17). In this section, we shall carry out this inverse construction explicitly in the case in which $u$ really is just a function of $t$, so that $d_M u = 0$.

The above assumption reduces (16) to

$$\tilde{\omega}_1 = \omega + t\omega'$$

for some closed (1,1)-forms $\omega, \omega'$ on $M$. Consider the real symmetric bilinear form $B$ defined by

$$\alpha \wedge \beta = \tfrac{1}{2} B(\alpha, \beta) \Omega \wedge \overline{\Omega} \tag{20}$$

on the space of 2-forms. Restricting $B$ to the subspace $\langle \omega, \omega' \rangle$ and diagonalizing, we may write

$$\tilde{\omega}_1 = (p + qt)\omega_0 + (r + st)\omega_1, \tag{21}$$

where

$$\omega_0 \wedge \omega_0 = -\tfrac{1}{2}\varepsilon \Omega \wedge \overline{\Omega}, \quad \omega_1 \wedge \omega_1 = \tfrac{1}{2} \Omega \wedge \overline{\Omega}, \quad \omega_0 \wedge \omega_1 = 0, \tag{22}$$

$\varepsilon$ is 0 or 1, and $p, q, r, s$ are constants satisfying

$$r + st > |p + qt| \tag{23}$$

to ensure the overall positivity of $\tilde{\omega}_1$. ¿From (21) and (17) we have

$$u = t\Big((r+st)^2 - (p+qt)^2\Big). \tag{24}$$

Note that $\varepsilon = 0$ in (22) if and only if $\omega_0 = 0$; to avoid redundancy we declare that $p = q = 0$ in this case.

The real symplectic forms $\omega_1, \omega_2, \omega_3$ satisfy the usual compatibility relations

$$\begin{aligned} \omega_i \wedge \omega_i &= \omega_j \wedge \omega_j, \\ \omega_i \wedge \omega_j &= 0, \quad i \neq j, \end{aligned} \tag{25}$$

extending (11). It is well known ([23, 29]) that they then determine a hyperkähler structure, consisting of

(i) complex structures $J_1, J_2, J_3$ satisfying $J_i \circ J_j + J_j \circ J_i = -2\delta_{ij}\mathrm{Id}$;
(ii) a Riemannian metric $g_0$ and associated Levi-Civita connection relative to which the $J_i$ are all orthogonal and parallel.

When $p^2 + q^2 > 0$, in addition to the hyperkähler structure, the $(1,1)$ form $\omega_0$ defines an almost-complex structure $I$ on $M$, such that $\omega_0(\,\cdot\,,\,\cdot\,) = g_0(I\,\cdot\,,\,\cdot\,)$ as in (3). It follows that $(g_0, I, \omega_0)$ is a Ricci-flat *almost-Kähler* metric on $M$, compatible with the opposite orientation to the one induced on $M$ by the hyperkähler structure $(\omega_1, \omega_2, \omega_3)$. The integrability of the almost-complex structure $I$ is equivalent to the flatness of the metric $g_0$, and this is the only possibility when $M$ is compact, see [31]. However, completely explicit local examples of 4-dimensional hyperkähler manifolds admitting a *non-integrable* almost-Kähler structure $I$ are now known [5, 7, 28].

**Theorem 2.** *Let $(M, g_0, \omega_1, \omega_2, \omega_3)$ be a hyperkähler 4-manifold. Let $r, s$ be real constants with $r + st > 0$ for any $t \in (a, b)$ with $a > 0$, and set*

$$\tilde{\omega}_1 = (r+st)\omega_1, \quad u = t(r+st)^2.$$

*Then the 3-form (18) defines a torsion-free $G_2$ structure on $(a,b) \times \mathbb{R}^2_{x,y} \times M'$ where $M'$ is a suitable open subset of $M$.*

*Suppose, furthermore, that $(M, g_0)$ admits an almost-Kähler structure $(\omega_0, I)$ compatible with the opposite orientation to the one induced by $(\omega_1, \omega_2, \omega_3)$. Let $p, q, r, s$ be real constants satisfying (23) for $t \in (a, b)$, $a > 0$, and set*

$$\tilde{\omega}_1 = (p+qt)\omega_0 + (r+st)\omega_1, \quad u = t\Big((r+st)^2 - (p+qt)^2\Big).$$

*Then (18) again defines a torsion-free $G_2$ structure on a manifold of the form $(a,b) \times \mathbb{R}^2_{x,y} \times M'$.*

**Remark 3.** It suffices to take $M'$ to be any contractible open subset of $M$, in which case the 1-forms $\xi_P$ and $\eta_N$ of Theorem 1 can always be defined on $P = \mathbb{R}^+_t \times M'$ and $N = \mathbb{R}^+_t \times \mathbb{R}_x \times M'$ respectively. However, as we shall see below, we can alternatively keep the 4-manifold $M$ fixed and think of the 1-forms $(\xi, \eta)$ in Theorem 1 as connection forms of a principal $\mathbb{T}^2$ bundle $W$ over $M$. Since

$$(d\xi, d\eta) = (q\omega_0 + s\omega_1, -\omega_2) \tag{26}$$

is the curvature of the principal connection, we obtain *integrality* constraints for the cohomological classes $[\frac{1}{2\pi}(q\omega_0 + s\omega_1)]$ and $[\frac{1}{2\pi}\omega_2]$ of $M$, in the sense that they must be contained in the image of the universal morphism $H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R})$.

Note also that in general only two of the four real parameters $(p,q,r,s)$ are effective in the sense that one can fix two of them, by rescaling $\varphi$ and $V$ on $Y$. Moreover, if $q = s = 0$, (18) corresponds to the product of a Calabi-Yau 6-manifold with $\mathbb{R}_x$, while generically the corresponding metric has holonomy equal to $G_2$ in accordance with Remark 2.

By way of proving Theorem 2, we shall describe the reverse construction in the case $(p,q,r,s) = (0,0,0,1)$ for simplicity, so that $u = t^3$. This is also the case in which the family $g(t)$ consists of *homothetic* hyperkähler metrics on $M$. The resulting metrics with holonomy $G_2$ were first described in [20] in the context of $\mathbb{T}^2$ bundles. To duplicate this situation, we assume that the 2-forms $\frac{1}{2\pi}\omega_1$ and $\frac{1}{2\pi}\omega_2$ are integral. This is always true locally, though if $M$ is compact (i.e. a K3 surface or a torus) the assumptions imply that $(M, J_3)$ is *exceptional* – its Picard number is maximal, see e.g. [9]. On any exceptional K3 surface, Yau's theorem implies the existence of hyperkähler metrics satisfying these integrality assumptions.

Let $P$ be the total space of the principal $S^1$ bundle over the hyperkähler 4-manifold $M$ classified by $[\frac{1}{2\pi}\omega_1]$, and $\xi$ a connection 1-form on $P$ such that $d\xi = \omega_1$ in accordance with (26). Attached to $P$ is also a principal $\mathbb{C}^*$ bundle over $M$, whose total space $N$ is the real 6-manifold manifold $N \cong \mathbb{R}^+_t \times P$. Pulling back forms to this new total space, we define on $N$ an almost-Hermitian structure $(h,\sigma,J)$ by

$$\begin{aligned} \sigma &= t\omega_1 + dt \wedge \xi \\ h &= tg_0 + t^{-3}\xi \otimes \xi + t^3 dt \otimes dt \end{aligned} \tag{27}$$

and

$$J\alpha = J_1\alpha,\ \alpha \in \Lambda^1 M, \qquad J\,dt = t^{-3}\xi.$$

The 2-form $\sigma$ is closed, satisfies (3), and it is easy to check that $J$ is integrable. Thus, $(h,\sigma,J)$ defines a Kähler structure on $N$.

Now let $U$ be vector field which is $h$-dual to $t^{-3}\xi$ so that $U$ is tangent to the fibers of $P$ and $\xi(U) = 1$; it follows that $U$ is the generator of the natural $S^1$ action on $N$, acting as a rotation on each fibre of $P$. Moreover, $U$ is a Hamiltonian isometry of the Kähler structure (27) and the corresponding momentum map is $-t$.

The integrality assumption for $\omega_2$ implies that there exists a principal $S^1$ bundle over $N$, classified by $[-\frac{1}{2\pi}\omega_2] \in H^2(N,\mathbb{Z})$. We denote by $Y$ the corresponding 7-dimensional total space and take $\eta$ to be a connection 1-form satisfying (12).

**Corollary 4.** *With the above assumptions, the* 3*-form*

$$\varphi = t\omega_1 \wedge \eta + dt \wedge \xi \wedge \eta + t^4\omega_3 \wedge dt + t\omega_2 \wedge \xi$$

*defines a torsion-free* $G_2$ *structure on* $Y$*. The corresponding 4-form is*

$$*\varphi = \omega_3 \wedge \xi \wedge \eta - t^3\omega_2 \wedge dt \wedge \eta + t^3\omega_1 \wedge dt \wedge \xi + \tfrac{1}{2}t^4\omega_1 \wedge \omega_1.$$

**Remark 4.** In the above situation, the Kähler structure $(h,\sigma,J)$ on $N$ belongs to the classes of metrics recently studied in [4] and [18]. In the terminology of the former, $(h,J)$ arises from a Hamiltonian 2-form of order 1 with one non-constant eigenvalue (equal to $t$) and one zero constant eigenvalue of multiplicity 2; the structure function $F(t)$ of $h$ is $t^{-1}$.

Moreover,

$$\sigma = t\omega_1 + dt\wedge\xi = d(t\xi) = d(t^4 Jdt) = dd^c(\tfrac{1}{5}t^5),$$

using the definition of $\xi$ after (12), and (26),(27). Not only then is $f(t)=\frac{1}{2}\log t$ a Ricci potential for $h$ (see Corollary 2), but $\frac{2}{5}t^5 = \frac{2}{5}e^{5f}$ acts as a Kähler potential for the same metric.

We now return to the general case of Theorem 2, when $p$ and $q$ are not both zero. Now the hyperkähler metrics $g(t)$ are not homothetic, though for the examples in [5, 7, 28] the family $g(t)$ is an isotopy (i.e. $g(t)$ are all isometric to an initial hyperkähler metric, under the flow of a vector field on $M$).

If the cohomology classes of $\frac{1}{2\pi}(q\omega_0+s\omega_1)$ and $\frac{1}{2\pi}\omega_2$ are integral, the $G_2$ structure is defined on the product of an interval and the principal $\mathbb{T}^2$ bundle $W$ over $M$, classified by $[\frac{1}{2\pi}(q\omega_0+s\omega_1)]$ and $[-\frac{1}{2\pi}\omega_2]$. The action of the 2-torus $\mathbb{T}^2$ is generated by the commuting vector fields $U=k(\,\cdot\,,t^{-2}\xi)$ and $V=k(\,\cdot\,,t^{-2}\eta)$, which preserve the $G_2$ structure.

Take $M=\mathbb{T}^4$ to be the 4-torus with a flat hyperkähler metric. There are three cases according to the signature of the bilinear form $B$ of (20) restricted to the 2-dimensional subspace generated by (26):

(i) $s>q$, so $B$ is positive definite and there is a flat metric $g_0$ on $\mathbb{T}^4$ with $d\xi, d\eta\in\Lambda^2_+$. Thus, there exists a basis $(e^i)$ of 1-forms on $\mathbb{T}^4$ such that $d\xi=\omega_1$ and $d\eta=-\omega_2$ with

$$\omega_1 = e^{14}+e^{23},\quad \omega_2 = e^{13}+e^{42},\quad \omega_3 = -(e^{12}+e^{34}),$$

reflecting the structure of the real Lie algebra $\mathfrak{h}$ associated to the complex Heisenberg group $H$. Then $W=\Gamma\backslash H$ is the Iwasawa manifold [1, 2].

(ii) $s=q$ so that $d\xi$ is a simple 2-form. In this case, we may assume that $d\xi=e^{24}$ and either $d\eta=e^{14}+e^{23}$ or $d\eta=e^{14}$. Then $W$ is a $\mathbb{T}^2$ bundle over $\mathbb{T}^4$ corresponding to one of two other nilmanifolds. The simplest holonomy $G_2$ metric in the first case is obtained by setting $(p,q,r,s)=(-1,1,1,1)$ so that $u=4t^2$. The resulting metric coincides with that described in Example 2 of [15, §4] (except that $t^2$ there has now become $t>0$).

(iii) $s<q$ and there is a basis with $d\xi=e^{14}-e^{23}$ and $d\eta=e^{14}+e^{23}$, so $\langle d\xi,d\eta\rangle=\langle d\tilde\xi,d\tilde\eta\rangle$ with $d\tilde\xi=e^{14}$ and $d\tilde\eta=e^{23}$. In this case, we may take $W$ to be a discrete quotient of $H_3\times H_3$, where $H_3$ is the real Heisenberg group.

**Example 1.** To make (i) more explicit, we may take coordinates $\lambda,\mu,\ell,m$ on $\mathbb{T}^4$ and fibre coordinates $x,y$ on $W$ so that

$$e^5 = -\eta = dx-\lambda d\ell+\mu dm,\quad e^6=\xi=dy-\mu d\ell-\lambda dm$$

are the corresponding connection 1-forms. The resulting $G_2$ metric

$$k = t^2(d\lambda^2+d\mu^2+d\ell^2+dm^2)+t^{-2}(dx-\lambda d\ell+\mu dm)^2+t^{-2}(dy-\mu d\ell-\lambda dm)^2+t^4dt^2$$

is defined on $(0,\infty)\times W$. It is shown in [20] to arise from an $SO(5)$ invariant $G_2$ metric on the total space of $\Lambda^2_+ \to S^4$ by a contraction of the isometry group.

The explicit form above makes it easy to compute the the Riemann tensor $R_{ijkl}$ of $k$. The package `GRTensor` (available from `http://grtensor.phy.queensu.ca`) was in fact used to verify that (a) the Ricci tensor $R^i{}_{jil}$ is zero, (b) the matrix $R^i{}_{jkl}$ with rows labelled by $i$ has rank 7 everywhere, and (c) the matrix $R^{ij}{}_{kl}$ with rows labelled by $(i,j)$ has rank 14 everywhere. Point (b) confirms that $k$ is irreducible and therefore has holonomy *equal* to $G_2$, which is consistent with (c). The same technique can be used to analyse metrics arising from (ii) and (iii).

## 4. Hypersurface structures

We shall now investigate the metrics with holonomy $G_2$ constructed in Theorem 2 by restricting them to hypersurfaces on which $t=\|\eta\|_k$ is constant *before* taking an $S^1$ quotient. These hypersurfaces correspond to the total spaces $W$ of the $\mathbb{T}^2$ bundles of Example 1.

We can write the 3-form (18) as

$$\varphi = (\tilde\omega_1 + dt\wedge\xi)\wedge\eta + t(u\omega_3\wedge dt + \omega_2\wedge\xi)$$

where $\xi,\eta$ are the corresponding connection 1-forms on $N, Y$ respectively. Since $\|dt\|_h^2 = u^{-1}$, it follows from (2) that $\|dt\|_k^2 = z^{-1}$ where $z = ut$, so

$$z = t^2\Big((r+st)^2-(p+qt)^2\Big) \tag{28}$$

The 1-form $z^{1/2}dt$ therefore has unit norm relative to the $G_2$ metric $k$, and we may write

$$z^{1/2}dt = d\tau \tag{29}$$

where

$$\tau = \int t\Big((r+st)^2-(p+qt)^2\Big)^{1/2}dt.$$

The important point here is that $u$ is constant as a function on $M$, and so $z$ is really just a function of $t$.

As a consequence,

$$\begin{aligned}\varphi &= \rho\wedge d\tau + \phi^+\\ *\varphi &= \phi^-\wedge d\tau + \tfrac12\rho\wedge\rho,\end{aligned} \tag{30}$$

where

$$\begin{aligned}\rho &= z^{1/2}\omega_3 + z^{-1/2}\xi\wedge\eta,\\ \phi^+ &= \tilde\omega_1\wedge\eta + t\omega_2\wedge\xi,\\ \phi^- &= t^{-1}z^{1/2}\omega_2\wedge\eta - t^2z^{-1/2}\tilde\omega_1\wedge\xi.\end{aligned} \tag{31}$$

Restricting to an interval where the function $t\mapsto\tau$ is bijective, let $Y_\tau$ denote the hypersurface of $Y$ for which $\tau$ has the constant value '$\tau$' (excusing the abuse of notation). Whereas $\sigma, \psi^+, \psi^-$ characterize the $SU(3)$ structure on $N$, we are using $\rho, \phi^+, \phi^-$ (a lexicographic shift) for the corresponding objects on $Y_\tau$.

The closure of the forms (30) is equivalent to asserting that

$$d\phi^+ = 0, \qquad d(\rho^2) = 0, \tag{32}$$

($\rho^2$ denotes $\rho \wedge \rho$) for every fixed value of $\tau$, and that the $SU(3)$ structures on $Y_\tau$ satisfy the equations

$$\frac{\partial \phi^+}{\partial \tau} = d\rho, \qquad \frac{\partial}{\partial \tau}(\tfrac{1}{2}\rho^2) = -d\phi^-. \tag{33}$$

An $SU(3)$ structure satisfying (32) is called *half-integrable* or *half-flat* in the formalism of [15].

To verify (33) directly in our situation, first observe that

$$d(\xi \wedge \eta) = (q\omega_0 + s\omega_1) \wedge \eta + \xi \wedge \omega_2$$

(recall (26)), and

$$\begin{aligned} \phi^+ &= (p\omega_0 + r\omega_1) \wedge \eta + t\,d(\xi \wedge \eta) \\ \tfrac{1}{2}\rho^2 &= \xi \wedge \eta \wedge \omega_3 + \tfrac{1}{2} z\omega_3 \wedge \omega_3. \end{aligned} \tag{34}$$

These equations explain the significance of the coordinates $(t, z)$ – the above forms are linear in $t$ and $z$ respectively. The reader may now check that (33) implies both (28) and (29); a constant of integration may be absorbed into the term $r^2 - p^2$.

Hitchin discovered that (33) leads to a Hamiltonian system in the symplectic vector space $\mathbf{V} \times \mathbf{V}^*$ where $\mathbf{V}$ is the space of exact 3-forms on the compact manifold $Y_\tau$, whose dual $\mathbf{V}^*$ can be identified with the space of exact 4-forms. The Hamiltonian function $H$ is derived from integrating volume forms determined algebraically by $\phi^+$ and $\rho^2$, and this also enables $\phi^-$ to be determined from $\phi^+$ in (31). Elements of $\mathbf{V}$ represent deformations of $\phi^+$ in a fixed cohomology class, and those of $\mathbf{V}^*$ deformations of $\rho^2$. Given a solution of (32) for $\tau = a$, a solution of (33) can then be found on some interval $(a, b)$ [21]. This approach also underlies some of the newly constructed metrics (such as [12]) with reduced holonomy.

The function $H$ is already implicit in our calculations above, which may be summarized in the following way:

**Proposition 2.** *The solution (31) can be expressed in the form $H = 0$ where the function $H = 2t\Big((r+st)^2 - (p+qt)^2\Big)^{1/2} - 2z^{1/2}$ satisfies*

$$\frac{dt}{d\tau} = -\frac{\partial H}{\partial z}, \qquad \frac{dz}{d\tau} = \frac{\partial H}{\partial t}.$$

It is an important consequence of the Kähler assumption that, for each choice $(p, q, r, s) \in \mathbb{R}^4$, there is only one valid solution curve in the $(t, z)$ plane.

**Example 2.** In the light of Proposition 2, a variant of (31) and Example 1 is provided by setting

$$\begin{aligned} \rho &= \pm z^{1/2}(e^{12} + e^{34}) + z^{-1/2} e^{56}, \\ \phi^+ &= \phi_0^+ + t\,d(e^{56}), \\ \phi^- &= \phi_0^- \pm t(e^5 \wedge de^5 + e^6 \wedge de^6), \end{aligned}$$

with

$$z = (t^2 - \tfrac{1}{2}H)^2. \tag{35}$$

We assume that $z^{1/2} > 0$, so that the orientation $\rho^3$ remains fixed. The notation is consistent with that of [1], with $de^5 = e^{13} + e^{42}$ and $de^6 = e^{14} + e^{23}$. Any value of the constant $H$ gives a valid solution and so a holonomy $G_2$ metric on the product $(a, b) \times W$ of some interval with the Iwasawa manifold $W$, though $\phi_0^\pm$ and the signs must be chosen to ensure compatibility (essentially (7)) and positive definiteness of the resulting metric.

The Figure plots the quartic curves (35) in the $(t, z)$ plane for various values of $H$; two pass through each point because of the sign ambiguity implicit in the definition of $H$.

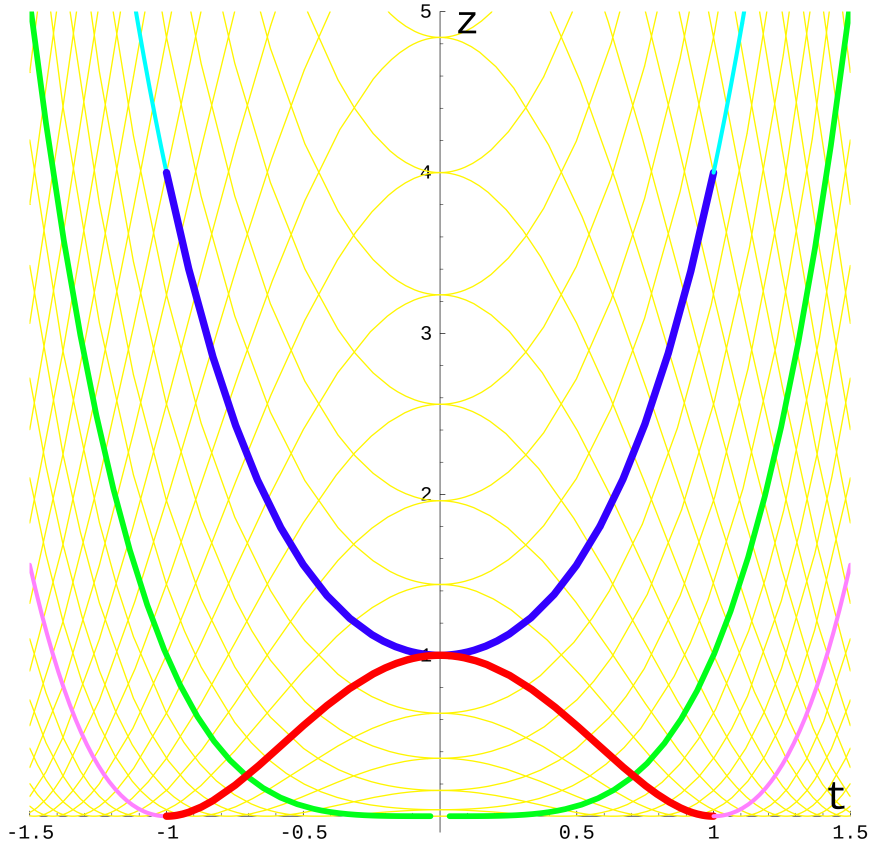
(i) $H = 2$ gives $z = (1 - t^2)^2$, including the bell-shaped segment. To satisfy (33) for $|t| < 1$ we need to choose the plus sign in $\phi^-$. We may then define the 3-forms $\phi_0^\pm$ by setting

$$\phi^+ + i\phi^- = i((e^5 + ie^6) + t(e^5 - ie^6)) \wedge d(e^5 + ie^6).$$

It follows that $\phi_0^+ + i\phi_0^-$ is a form of type $(3, 0)$ relative to the standard complex structure on $W$, and taking $z = 1$ and the plus sign in $\rho$ determines a compatible Hermitian metric via (3). As one flows away from the point $(0, 1)$, the almost-complex structure induced on the hypersurface becomes non-integrable; $\rho, \phi^\pm$ degenerate simultaneously when one reaches $t = \pm 1$. Furthermore, we may take $\tau = \frac{1}{3}t^3 + t$, a cubic equation with solution

$$t = \alpha^{1/3} - \alpha^{-1/3}, \quad 2\alpha = 3\tau + (9\tau^2 - 4)^{1/2},$$

in contrast with the Kähler scheme in which $\tau = c + \frac{1}{2}rt^2 + \frac{1}{3}st^3$ in the case $p = q = 0$.




Figure

(ii) $H=-2$ gives $z=(t^2+1)^2$ (the curve above and touching the bell). This requires the minus sign in $\phi^-$ and provides a different deformation of the standard Hermitian structure on $W$. Indeed, $\phi_0^+ + i\phi_0^-$ is modified by the addition of a form of type $(1,2)$ rather than $(2,1)$, and the resulting almost-complex structure is undefined when $|t|$ reaches 1. However, $\rho$ remains non-degenerate for all $t$, and this corresponds to a different singular behaviour of the $G_2$ metric.

(iii) $H=0$ and $z=t^4$ (with the flattened base) requires $\phi_0^\pm=0$ and the minus signs in $\phi^-,\rho$, reproducing exactly the solution of Example 1. The almost-complex structure induced on $W$ by $\phi^+$ is *constant* and was first singled out for study in [2], where it is called $J_3$. The 2-form $\rho$ degenerates only for $t=0$, and the resulting metric has the advantage of being 'half-complete'.

We conclude this section by providing an explanation of why the system (33) often reduces to one variable. Let $\phi^+$ be an invariant closed 3-form with stabilizer $SL(3,\mathbb{C})$ on either of the four nilmanifolds $\Gamma\backslash H$ described just before Example 1. Let $\phi^-$ be the unique 3-form for which $\phi^+ + i\phi^-$ has type (3,0) relative to the almost-complex structure $J$ determined by $\phi^+$. Let $\ker d$ denote the space of invariant closed 1-forms (equivalently, the kernel of the natural mapping $d:\mathfrak{h}^*\to\bigwedge^2\mathfrak{h}^*$).

**Lemma 2.** *In the above situation, $d\psi_+=0$ implies that $J(\ker d)=\ker d$.*

This can be proved by generalizing an argument in the proof of [26, Theorem 1.1], which draws the same conclusion if $J$ is integrable. In view of the structure equations for the Lie algebra $\mathfrak{h}$, the condition that $J$ leave $\ker d$ invariant is equivalent to asserting that $d\phi^-$ belong to the 1-dimensional space

$$\langle e^{1234}\rangle = \bigwedge{}^4(\ker d),$$

but it is this fact that simplifies the equations.

## 5. Solutions varying on $M$

As another ramification of the evolution equations for $(\tilde\omega_1 = \tilde\omega_1(t), u = u(t))$, we consider the case when $(M, g_0, \omega_1, \omega_2, \omega_3)$ is a hyperkähler manifold, and

$$\tilde\omega_1 = \omega_1 - \tfrac{1}{2} d_M d^c_M G,$$
$$2u = G''$$

for a smooth function $G$ on $(a, b) \times M$, where $'$ continues to denote $\partial/\partial t$. We assume here that $\tilde\omega_1$ is a positive definite (1,1)-form on $(M, J_1)$ for each fixed $t$ in the interval $(a, b)$. Whilst (16) is automatically satisfied, (17) becomes

$$2t\mathcal{M}(G) = G'' \tag{36}$$

where $\mathcal{M}$ denotes the complex Monge-Ampère operator on $(M, g_0, J_1)$, defined by

$$\Big(\omega_1 - \tfrac{1}{2} d_M d^c_M f\Big)^{\wedge 2} = \mathcal{M}(f)\omega_1 \wedge \omega_1,$$

for all $f \in C^\infty(M)$. Note that the 1-form $\xi_P$ of Theorem 1 is automatically defined on $P = \mathbb{R}^+_t \times M$ by

$$\xi_P = -\tfrac{1}{2} d^c_M(G'). \tag{37}$$

Since the hyperkähler metric $g_0$ is Ricci-flat and the $\omega_i$'s are parallel 2-forms, there exists a real-analytic structure on $M$, compatible with $(g_0, J_1, \omega_1, \omega_2, \omega_3)$. Thus, applying the Cauchy-Kowalewski theorem, one obtains a general existence result.

**Corollary 5.** *Let $(M, g_0, \omega_1, \omega_2, \omega_3)$ be a hyperkähler, real-analytic 4-manifold and $J_1$ be the Kähler structure compatible with $\omega_1$. Suppose that $G^0, G^1$ are real-analytic functions on $M$ with $\omega_1 - \frac{1}{2} d_M d^c_M G^0$ positive definite with respect to $J_1$. Then there exist a real number $a > 0$ and a real-analytic solution $G(t, \cdot\,)$ of (36) defined on $(0, a) \times M$ with $G(0) = G^0$, $G'(0) = G^1$, and such that $\omega_1 - \frac{1}{2} d_M d^c_M G$ is positive definite for any $t \in (0, a)$. Thus, $\tilde\omega_1 = \omega_1 - \frac{1}{2} d_M d^c_M G$ and $u = \frac{1}{2} G''$ define, via Theorem 1, a torsion-free $G_2$ structure on a manifold of the form $(0, a) \times \mathbb{R}^2_{x,y} \times M'$ where $M'$ is a suitable open subset of $M$.*

In the above construction $M'$ should be taken so as to solve $d\eta_N = -\omega_2$ for a 1-form $\eta_N$ on $N = \mathbb{R}^+_t \times \mathbb{R}_x \times M'$ (see Theorem 1 and (37)). Alternatively, we may assume that the cohomology class $[\frac{1}{2\pi}\omega_2]$ of $M$ is integral and $\eta$ is a principal connection of the principal $S^1$ bundle $Q$ over $M$, classified by $[-\frac{1}{2\pi}\omega_2]$. In this case Corollary 5 produces examples of torsion-free $G_2$ structures with a $\mathbb{R} \times S^1$ symmetry on $Y = (0, a) \times \mathbb{R}_x \times Q$. One has no control over the real number $a$.

It is tempting to spot some special solutions to (36), by reducing the problem to a linear (elliptic) equation. This can be done by assuming that for each $t > 0$ the function $G$ generates a complex *Monge-Ampère foliation* on $(M, J_1)$ (see e.g. [10]), meaning that

$$d_M d_M^c G \wedge d_M d_M^c G = 0$$

and $d_M d_M^c G$ has constant rank. (The integral curves of $d_M d_M^c G$ then foliate $M$ by complex submanifolds.) The point is that in this case we have

$$\mathcal{M}(G) = 1 + \tfrac{1}{2}\Delta G,$$

where $\Delta$ is the Riemannian Laplacian of the hyperkähler metric $g_0$.

The above situation appears in particular when $(M, J_1)$ admits a holomorphic $\mathbb{C}$ action and we look for equivariant solutions of (36), or when $(M, J_1)$ admits a holomorphic fibration $p : M \to C$ over a complex curve $C$ and we look for solutions of the form $G \circ p$ where for each $t$, $G$ is a function on $C$.

Consider finally the flat hyperkähler metric $g_0$ on $(M, J_1) = \mathbb{C}^2 \cong \mathbb{R}^4$, determined by the 2-forms

$$\omega_1 = \tfrac{1}{2}i(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2), \quad \omega_2 = \mathfrak{Re}(dz_1 \wedge dz_2), \quad \omega_3 = \mathfrak{Im}(dz_1 \wedge dz_2),$$

where $z_1 = \lambda + i\mu$ and $z_2 = \ell + im$ are the canonical coordinates of $\mathbb{C}^2$. Letting $G(t, z_1)$ be a function on $\mathbb{C}^2$ which does not depend on $z_2$, the equation (36) reduces to

$$G'' + t\left(\frac{\partial^2 G}{\partial \lambda^2} + \frac{\partial^2 G}{\partial \mu^2}\right) = 2t.$$

Separable solutions are given by $G(t, \lambda, \mu) = \frac{1}{3}t^3 + H(\lambda, \mu)K(t)$, where

$$\frac{\partial^2 H}{\partial \lambda^2} + \frac{\partial^2 H}{\partial \mu^2} + cH = 0,$$

$c$ is a constant and $K$ a solution of the Airy equation $K'' = ctK$ (equivalently, $L = K^{-1}K'$ satisfies the Riccati equation $L' + L^2 = ct$).

**Example 3.** Taking $H$ to be periodic in $\lambda, \mu$ (which requires $c > 0$) yields solutions of (36) defined on $(0, a) \times \mathbb{T}^4$. One such example is obtained by taking $H = \sin\lambda$ (so $c = 1$), and

$$K = \mathrm{Ai}(t) = \tfrac{1}{3}t^{1/2}(J_{1/3}(\zeta) + J_{-1/3}(\zeta)), \quad \zeta = \tfrac{2}{3}it^{3/2}.$$

Setting $f = 1 + \mathrm{Ai}(t)\sin\lambda$, the resulting $G_2$ metric

$$t(f\,d\lambda^2 + f\,d\mu^2 + d\ell^2 + dm^2) + f^{-1}(dx - \mathrm{Ai}'(t)\cos\lambda\, d\mu)^2 + t^{-2}(dy - \lambda d\ell + \mu dm)^2 + t^2 f\, dt^2$$

is Ricci-flat and irreducible. Since $\mathrm{Ai}(t) \to 0$ as $t \to \pm\infty$, the above metric is asymptotic to a constant solution with $u = t$ and holonomy equal to $SU(3)$ (see Remark 3). However, the above construction can be easily modified to provide explicit deformations of the non-trivial metrics constructed in §3.

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